Homotopic Routing Methods

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1. Introduction

The problem:

(1) Given: – a graph $G = (V, E)$,
– pairs $r_1, s_1, \ldots, r_k, s_k$ of vertices of G;
find: – pairwise disjoint paths $P_1, \ldots, P_k$ in G, where P_i connects r_i and s_i (for $i = 1, \ldots, k$),

is NP-complete, even for planar graphs, both in the vertex-disjoint and in the edge-disjoint case (Lynch [26]). In some special cases, however, there is a polynomial-time method for (1). These cases usually also give rise to a theorem characterizing the existence of a solution as required.

Moreover, if G is planar, one can design a heuristic or enumerative approach based on the topology of the plane. It amounts to selecting a, possibly small, set of faces $I_1, \ldots, I_p$ of G so that each of the vertices $r_1, s_1, \ldots, r_k, s_k$ is incident with at least one of these faces. Next we choose (or enumerate) for each pair r_i, s_i a curve C_i in $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ connecting r_i and s_i. Our problem then is to find pairwise disjoint paths $P_1, \ldots, P_k$ so that P_i is homotopic to C_i in the space $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$. If such P_i are found, we have solved our original problem. Otherwise, we choose other curves C_i (i.e., representing other homotopies), and try again.

So in this approach (proposed by Pinter [31]) we must solve the following problem:

(2) Given: – a planar graph $G = (V, E)$, embedded in $\mathbb{R}^2$,
– faces $I_1, \ldots, I_p$,
– curves $C_1, \ldots, C_k$ with end points on the boundary of $I_1 \cup \ldots \cup I_p$,
find: – pairwise disjoint simple paths $P_1, \ldots, P_k$ in G where P_i is homotopic to C_i in $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$, for $i = 1, \ldots, k$.

This problem also emerges from Robertson and Seymour's work on graph minors [34]. It turns out that this problem can be solved in polynomial time in the vertex-disjoint case. The edge-disjoint case appears to be more difficult (due to the fact that the curves $C_1, \ldots, C_k$ then can be quite wild). In fact, Kaufmann and Maley

[17] recently showed that the edge-disjoint version is NP-complete. In some special cases, a polynomial-time algorithm for the edge-disjoint case has been found.

In this paper we give a survey of the results and methods for problems (1) and (2). We moreover describe some links with problems on disjoint circuits in graphs on compact surfaces, and on disjoint trees of given homotopies.

Some Conventions and Terminology

By an *embedding* of a graph $G = (V, E)$ in the plane or any other surface, we mean an embedding without intersecting edges. When speaking of a planar graph, we implicitly assume it to be embedded in the plane. We identify an embedded graph with its topological image. Edges are considered as open curves, and faces as open regions. By bd (...) we denote the boundary of ... . An $r - s$*—path* is a path from r to s.

2. Vertex-Disjoint Paths and Trees

As mentioned, the problem:

(3) Given: – a graph $G = (V, E)$,
– pairs $r_1, s_1, \ldots, r_k, s_k$ of vertices of G,
find: – pairwise vertex-disjoint paths $P_1, \ldots, P_k$ in G, where P_i connects r_i and s_i (for $i = 1, \ldots, k$),

is NP-complete (Lynch [26]). On the other hand, Robertson and Seymour [35] showed:

Theorem 1. *For each fixed k, there is a polynomial-time algorithm for* (3).

In fact, the algorithm has running time $O(|V|^2 \cdot |E|)$, but the constant depends heavily on k. For details, see also Robertson and Seymour [36].

For the special case of planar graphs, there are some further polynomial-time methods. Clearly, a necessary condition for planar G is:

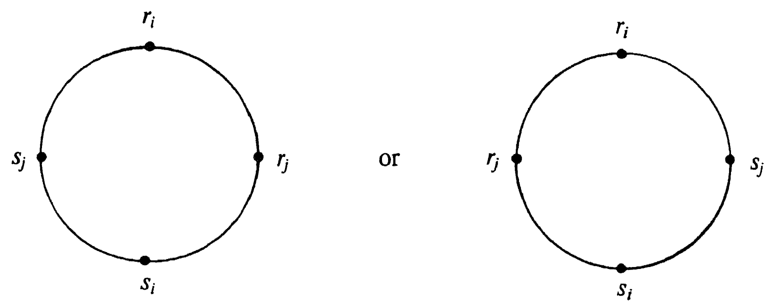
(4) (*Cut condition*) for each closed curve D in $\mathbb{R}^2$, the number of intersections with G is at least the number of pairs r_i, s_i separated by D.

Here D *separates* r_i, s_i if each curve connecting r_i and s_i intersects D. Obviously, in (4) we may restrict D to closed curves intersecting G only in vertices of G, and not in edges.

Robertson and Seymour [33] observed that there is an easy algorithm for (3) in the case where G is planar, and $r_1, s_1, \ldots, r_k, s_k$ all lie on the boundary of one face I. In that case a necessary condition is:

(5) (*Cross-freedom condition*) no two pairs r_i, s_i and r_j, s_j are crossing.

Here r_i, s_i and r_j, s_j are said to *cross* if r_i, s_i, r_j, s_j are all distinct and r_i, r_j, s_i, s_j occur cyclically (clockwise or anti-clockwise) around the boundary of I:



ıeorem 2. *If G is planar and $r_1, s_1, \ldots, r_k, s_k$ all are on the boundary of one face problem* (3) *is solvable in polynomial time.*

oof. Without loss of generality, $r_i \neq s_i$ for all i. We first check if $r_1, s_1, \ldots, r_k, s_k$; all distinct and if the cross-freedom condition holds. The cross-freedom ndition implies that there exists a pair r_i, s_i so that at least one of the two $r_i - s_i$ ths along the boundary of I does not contain any r_j or $s_j (j \neq i)$. Withc ;s of generality, $i = 1$. Let Q_1 be this path. Now if (3) has a solution, ther e with $P_1 = Q_1$ (as in any solution of (3), path P_1 can be "pushed" aga ; boundary of I). Leaving out the vertices in Q_1 from G, together with ges incident to them, we obtain a graph G'. We next solve problem (3) . , $r_2, s_2, \ldots, r_k, s_k$. If we find paths $P_2, \ldots, P_k$ then $P_1, P_2, \ldots, P_k$ form a solution ι ; original problem. Otherwise, (3) has no solution. □

fact this algorithm also easily implies the following theorem:

ıeorem 3. *Let G be planar, so that $r_1, s_1, \ldots, r_k, s_k$ are all on the boundary of one :e. Then* (3) *has a solution if and only if the cut condition and the cross-freedom ıdition holds.*

In fact, the following generalization of Theorem 2 follows from the homotopic proach to be described in Section 5 below (see Theorem 34):

ıeorem 4. *For each fixed p there exists a polynomial-time algorithm for problem , whenever G is planar so that $r_1, s_1, \ldots, r_k, s_k$ can be covered by the boundaries at most p faces.*

We conjecture that also the following holds:

ınjecture. Problem (3) is solvable in polynomial time whenever the graph $:= (V, E \cup \{\{r_1, s_1\}, \ldots, \{r_k, s_k\}\})$ is planar.

here the pairs $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ are now edges in H, which edges we may ;ume to form a matching in H.

:tension to Disjoint Trees

ıere is a direct extension of the above results to trees instead of paths. Consider ; problem:

(7) Given: – a graph $G = (V, E)$,
– subsets $W_1, \dots, W_k$ of V,
find: – pairwise vertex-disjoint trees $T_1, \dots, T_k$ where T_i covers W_i (for $i = 1, \dots, k$).

Again this problem is NP-complete (as it generalizes (3)). For planar graphs we can proceed similarly to above. Again a necessary condition is:

(8) (*Cut condition*) for each closed curve D in $\mathbb{R}^2$, the number of intersections with G is at least the number of W_i separated by D.

Here D *separates* W_i if D separates at least two points in W_i.

If all points in $W_1 \cup \dots \cup W_k$ are on the boundary of one face I, there is the following necessary condition:

(9) (*Cross-freedom condition)* no two sets W_i and W_j are crossing.

Here W_i and W_j are said to *cross* if W_i contains two points r', s' and W_j contains two points r'', s'' so that the pairs r', s' and r'', s'' cross.

Now the following two theorems extend Theorems 2 and 3:

Theorem 5. *If G is planar, and all vertices in $W_1 \cup \dots \cup W_k$ are on the boundary of one face I, then problem* (7) *can be solved in polynomial time.*

Theorem 6. *Let G be planar, so that all points in $W_1 \cup \dots \cup W_k$ are on the boundary of one face. Then problem* (7) *has a solution if and only if the cut condition and the cross-freedom condition hold.*

Again, the following generalization of Theorem 5 follows from the homotopic approach to be described in Section 5 below:

Theorem 7. *For each fixed p there exists a polynomial-time algorithm for problem* (7), *whenever G is planar so that the vertices in $W_1 \cup \dots \cup W_k$ can be covered by the boundaries of at most p faces.*

The conjecture above can be extended as follows. Let $u_1, \dots, u_k$ be new (abstract) vertices. Let F be the set of all pairs $\{u_i, w\}$ where $i \in \{1, \dots, k\}$ and $w \in W_i$. We conjecture:

Conjecture. Problem (7) is solvable in polynomial time whenever the graph $H := (V \cup \{u_1, \dots, u_k\}, E \cup F)$ is planar.

3. Edge-Disjoint Paths and Multicommodity Flows

We now turn to the edge-disjoint case. Consider the problem:

(10) Given – a graph $G = (V, E)$,
– pairs $r_1, s_1, \dots, r_k, s_k$ of vertices of G,
find: – pairwise edge-disjoint paths $P_1, \dots, P_k$ in G where P_i connects r_i and s_i (for $i = 1, \dots, k$).

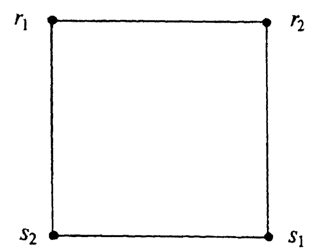
It is not difficult to see that Robertson and Seymour's theorem (Theorem 1 above) implies:

Theorem 8. *For each fixed k, there exists a polynomial-time algorithm for problem* (10).

This follows by considering the line-graph of G. In general however, problem (10) is NP-complete, even for planar G.

Again, a necessary condition for (10) is:

(11) (*Cut condition*) for each $X \subseteq V : |\delta(X)| \geq |\rho(X)|$.

(12) 

Here $\delta(X)$ denotes the set of edges with exactly one end point in X. By $\rho(X)$ we denote the set of those $i \in \{1,\ldots,k\}$ for which exactly one of r_i and s_i belongs to X.

As is well-known, Menger's theorem [27] states that the cut condition is also sufficient if $r_1 = \ldots = r_k$ and $s_1 = \ldots = s_k$. We leave it as an exercise to derive from this that the cut condition is sufficient if we require only $r_1 = \ldots = r_k$.

However, in the general case it is not a sufficient condition, as is shown by the simple example of (12) above.

So one may not hope for many more interesting cases where the cut condition suffices.

It turns out however that one more condition (which is clearly *not* a necessary condition) is quite powerful:

(13) (*Parity condition*) for each vertex v of G, the number

$$|\delta(\{v\})| + |\rho(\{v\})|$$

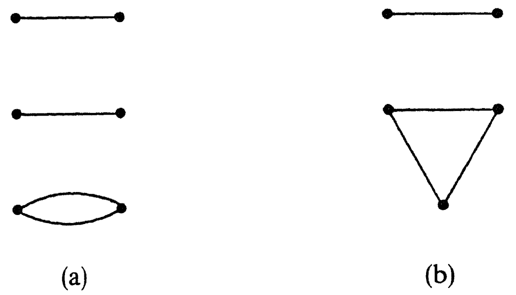
is even.

In particular, every vertex not in $\{r_1, s_1, \ldots, r_k, s_k\}$ should have even degree. This is why cases satisfying (13) sometimes are called *eulerian*.

The following is a theorem of Lomonosov [22, 23, 24] (extending earlier results of Hu [11], Rothschild and Whinston [37, 38], Dinits [1], Papernov [30] and Seymour [53] (cf. Lovász [25], Seymour [52])):

Theorem 9. *The cut condition implies that problem* (10) *has a solution, in case the parity condition holds and* $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ *do not contain four pairs forming one of the following configurations:*

(14)



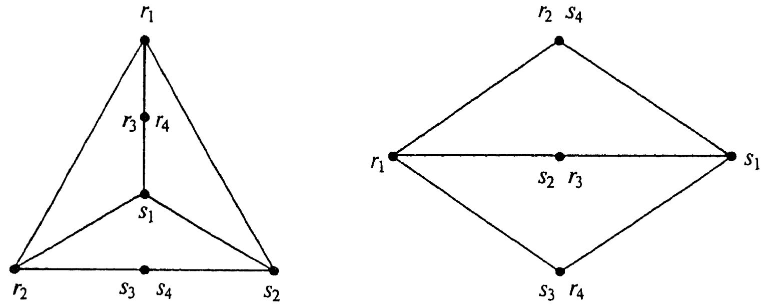
For a proof, see also Frank [6].

It is not difficult to see that excluding (14)(a) and (b) is equivalent to the condition that the graph on $\{r_1, s_1, \ldots, r_k, s_k\}$ with edges $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ (possibly parallel) is

(15) either (i) the complete graph K_4 (possibly with parallel edges), or (ii) the circuit C_5 (possibly with parallel edges), or (iii) the union of two stars (possibly with parallel edges), or (iv) a graph consisting of three disjoint edges.

The following examples show that the condition in Theorem 9 is in a sense tight:

(16)



Theorem 9 has the following implication for multicommodity flows. For any "demand" function $d : \{1, \ldots, k\} \longrightarrow \mathbb{Q}_+$ and any "capacity" function $c : E \longrightarrow \mathbb{Q}_+$, let a *multicommodity flow* be a system of paths $P_{11}, \ldots, P_{1t_1}, P_{21}, \ldots, P_{2t_2}, \ldots, P_{k1}, \ldots, P_{kt_k}$, together with a system of rationals $\lambda_{11}, \ldots, \lambda_{1t_1}, \lambda_{21}, \ldots, \lambda_{2t_2}, \ldots, \lambda_{k1}, \ldots, \lambda_{kt_k} \geq 0$ satisfying:

(17) (i) $\sum_{j=1}^{t_i} \lambda_{ij} = d_i \qquad (i = 1, \ldots, k),$

$$\text{(ii)} \quad \sum_{i=1}^{k}\sum_{j=1}^{t_i} \lambda_{ij}\mathscr{X}^{P_{ij}}(e) \le c(e) \qquad (e \in E).$$

Here $\mathscr{X}^P(e)$ denotes the number of times P passes e.

If the λ_{ij} are integral, we say that the multicommodity flow is *integral.* If the λ_{ij} are half-integral, we say that the multicommodity is *half-integral.* If $d_i = 1$ for all i and $c(e) = 1$ for all e, we call a multicommodity flow a *fractional* solution to problem (10). Indeed, an integral multicommodity flow then corresponds to a solution to (10).

Again we have a cut condition necessary for the existence of a multicommodity flow (given a demand function d and a capacity function c):

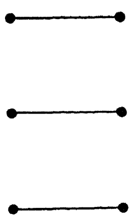
(18) (*Cut condition*) for each $X \subseteq V$: $\sum_{e\in\delta(X)} c(e) \ge \sum_{i\in\rho(X)} d_i$.

Note that there are the following implications:

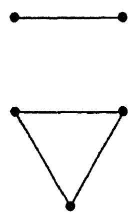
(19) $\exists$ integral multicommodity flow $\Longrightarrow$
$\exists$ half-integral multicommodity flow $\Longrightarrow$
$\exists$ multicommodity flow $\Longrightarrow$
cut condition.

Now Theorem 9 implies that in some cases we can reverse the implications, as was shown by Papernov [30] (forming an extension of Ford and Fulkerson's max-flow min-cut theorem [5]). Consider the property

(20) $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ do not contain one of the following configurations:



or



Theorem 10. *If d and c are integral-valued, and condition* (20) *is satisfied, then the cut condition* (18) *is equivalent to the existence of a half-integral multicommodity flow.*

This can be derived from Theorem 9 by replacing each edge e of G by $2c(e)$ parallel edges, and each pair $\{r_i, s_i\}$ by $2d_i$ "parallel" pairs.

Theorem 11. *If* (20) *is satisfied, then the cut condition is equivalent to the existence of a multicommodity flow.*

This can be seen by multiplying d and c by some natural number K so that Kd and Kc are integral, and next by applying Theorem 10.

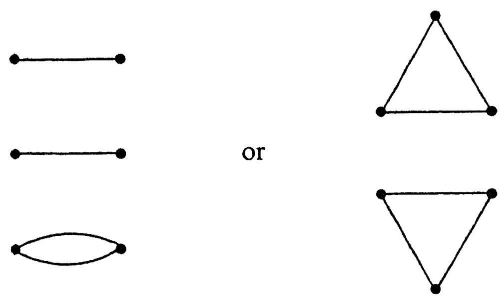
This theorem is tight in the sense that if (20) is not satisfied, there exists a graph G, a demand function d and a capacity function c for which the cut condition is satisfied, but no multicommodity flow as required exists - this can be derived directly from the examples (16).

If $d \equiv 1$ and $c \equiv 1$, Theorem 10 reduces to:

Theorem 12. *If* (20) *is satisfied, then the cut condition* (11) *is equivalent to the existence of a half-integral solution to problem* (10).

Karzanov [16] gave an extension of part of this result. Consider the property:

(21) $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ do not contain one of the following configurations:



It can be checked easily that this means that the graph on $\{r_1, s_1, \ldots, r_k, s_k\}$ with edges $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ is:

(22) either (i) the complete graph K_5 (possibly with parallel edges), or (ii) the union of a triangle and a star (possibly with parallel edges), or (iii) the union of two stars (possibly with parallel edges), or (iv) the graph consisting of three disjoint edges.

Karzanov showed:

Theorem 13. *If* (21) *holds and the parity condition holds, then the existence of a fractional solution to* (10) *implies the existence of a solution to* (10).

Again this implies:

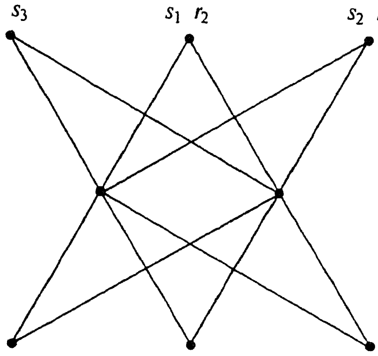
Theorem 14. *If* (21) *holds, then* (10) *has a half-integral solution if and only if* (10) *has a fractional solution.*

Example (23) shows that it is necessary to exclude the second configuration in (21).

In this example a fractional solution exists, but no integral solution. It is not known to me if also the first configuration in (21) must be excluded. In

[24], Lomonosov gives an example showing that it is necessary to require that $\{r_1, s_1\}, \ldots, \{r_k, s_h\}$ do not contain 38 pairs, covering 6 points, so that they fall apart in three sets of parallel edges, of sizes 2, 18 and 18, respectively.

(23)



Duality

Some of the results above have a "dual" counterpart, in terms of packing of cuts as was noticed by Karzanov [14] and Seymour [51]. Consider the convex cone K in $\mathbb{R}^k \times \mathbb{R}^E$ consisting of all vectors $(d;c)$ for which (17) has a solution $\lambda_{ij} \geq 0$. So K is the convex cone generated by all vectors

(24) (i) $(\varepsilon_i; \mathscr{X}^P)$ $(i = 1, \ldots, k; P$ is an $r_i - s_i$ path),
(ii) $(\mathbf{0}; \varepsilon_e)$ $(e \in E)$.

Here ε_i denotes the i-th unit basis vector in $\mathbb{R}^k$, and ε_e denotes the e-th unit basis vector in $\mathbb{R}^E$. By χ^P we denote the function in $\mathbb{R}^E$ given by $\chi^P(e) :=$ the number of times P passes e.

Now the content of Theorem 11 is that, if (20) is satisfied, then K is exactly the cone of all vectors which have nonnegative inner product with all vectors:

(25) (i) $(-\mathscr{X}^{\rho(X)}; \mathscr{X}^{\delta(X)})$ $(X \subseteq V)$,
(ii) $(\varepsilon_i; \mathbf{0})$ $(i = 1, \ldots, k)$,
(iii) $(\mathbf{0}; \varepsilon_e)$ $(e \in E)$.

Here $\mathscr{X}^{\rho(X)}$ and $\mathscr{X}^{\delta(X)}$ denote the incidence vectors of $\rho(X)$ and $\delta(X)$, respectively.

Now by duality (Farkas' lemma), the convex cone generated by the vectors (25) is exactly equal to the set of vectors having nonnegative inner product with all vectors (24) (if (20) is satisfied). In fact, it is equivalent to the following:

Theorem 15. *Let* (20) *be satisfied. Then there exist cuts* $\delta(X_1), \ldots, \delta(X_t)$ *and rationals* $\mu_1, \ldots, \mu_t \geq 0$ *so that:*

(26) (i) $dist_G(r_i, s_i) = \sum(\mu_j \mid i \in \rho(X_j))$ (for each $i = 1, \ldots, k$),

(ii) $\sum_{j=1}^{t} \mu_j \mathscr{X}^{\delta(X_j)}(e) \leq 1$ (for each $e \in E$).

Here $dist_G(r, s)$ denotes the distance between r and s in G. To derive Theorem 15 note that the vector

(27) $(-dist_G(r_1, s_1), \ldots, -dist_G(r_k, s_k); 1, \ldots, 1)$

has nonnegative inner product with all vectors in (24). Hence it can be written as a nonnegative linear combination of vectors in (25), yielding cuts $\delta(X_j)$ and rationals μ_j as required.

Now Karzanov [15] showed that if G is bipartite, we can take the μ_j integral. That means:

Theorem 16. *Let G be bipartite, and $r_1, s_1, \ldots, r_k, s_k$ be vertices of G so that* (20) *is satisfied. Then there exist pairwise disjoint cuts $\delta(X_1), \ldots, \delta(X_t)$ so that for each $i = 1, \ldots, k$:*

(28) $dist_G(r_i, s_i) =$ the number of cuts $\delta(X_j)$ separating r_i and s_i.

Here $\delta(X)$ is said to *separate* r and s if X contains exactly one of r and s. Theorem 16 extends theorems of Hu [12] and Seymour [50] for the case $k = 2$.

Theorem 16 implies:

Theorem 17. *The μ_j in Theorem 15 can be taken from $\{\frac{1}{2}, 1\}$.*

This follows by replacing each edge of G by two edges in series, thus making a bipartite graph.

For a short proof of some of the results in this section, see [47].

4. Edge-Disjoint Paths in Planar Graphs

Although the forbidden configurations given in Section 3 are "tight", there are more cases where the cut condition suffices, if we restrict G to planar graphs. Again, we consider problem (10). So we have a graph $G = (V, E)$ and pairs $r_1, s_1, \ldots, r_k, s_k$ of vertices, and we ask for pairwise edge-disjoint paths $P_1, \ldots, P_k$, where P_i connects r_i and s_i $(i = 1, \ldots, k)$.

A basic result due to Okamura and Seymour [29] requires the following property for planar G :

(29) G has a face I so that $r_1, s_1, \ldots, r_k, s_k$ all belong to the boundary of I.

Theorem 18. *Let G be planar so that* (29) *is satisfied. Moreover, let the parity condition* (13) *hold. Then* (10) *has a solution if and only if the cut condition holds.*

For a proof we refer to Frank [6].

In fact, Okamura [28] showed that condition (29) can be weakened to:

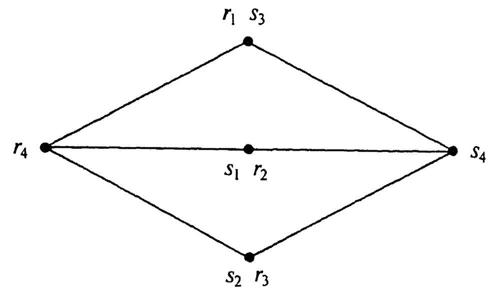
(30) G has faces I_1, I_2 so that for each $i = 1, \ldots, k : r_i, s_i \in bd(I_1)$ or $r_i, s_i \in bd(I_2)$.

Theorem 19. *Let G be planar so that* (30) *is satisfied. Moreover, let the parity condition* (13) *hold. Then* (10) *has a solution if and only if the cut condition holds.*

Also a proof of this theorem is given in Frank [6].

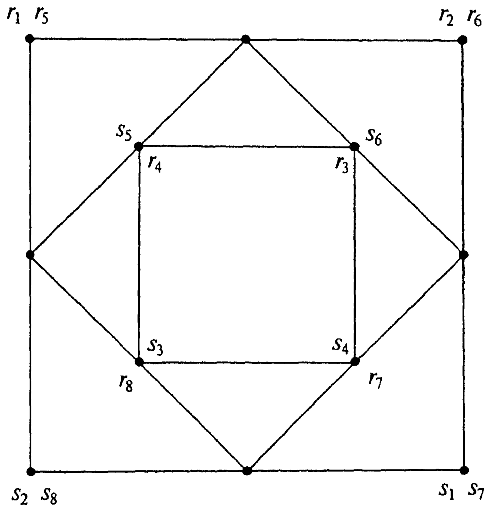
One may not allow in Okamura's theorem "mixed pairs", i.e. pairs r_i, s_i with $r_i \in bd(I_1)$ and $s_i \in bd(I_2)$. Neither can one extend the theorem to more than two faces. These facts are shown by the following example:

(31)



In fact, in this example not even a fractional solution exists. The following example (Hurkens, Schrijver and Tardos [13]), with mixed pairs, satisfying the parity condition, has a fractional solution, but no integral solution:

(32)



In [46] we showed that in a particular case of mixed pairs the cut condition suffices. Let I_1 and I_2 be two faces of G, where I_1 is (without loss of generality) the unbounded face. Let $r_1, s_1, \ldots, r_k, s_k$ be vertices so that:

(33) $r_1, \ldots, r_k$ are on $bd\ (I_1)$ in clockwise order,
$s_1, \ldots, s_k$ are on $bd\ (I_2)$ in anti- clockwise order.

Theorem 20. *Let G be planar so that* (33) *is satisfied. Moreover, let the parity condition* (13) *hold. Then* (10) *has a solution if and only if the cut condition holds.*

Example (31) also shows that we cannot allow $r_1, \ldots, r_k$ and $s_1, \ldots, s_k$ to occur both in clockwise order on $bd(I_1)$ and $bd(I_2)$, respectively.

Seymour [54] considered the following property:

(34) the graph $(V, E \cup \{\{r_1, s_1\}, \ldots, \{r_k, s_k\}\})$ is planar.

Theorem 21. *Let* (34) *and the parity condition* (13) *be satisfied. Then* (10) *has a solution if and only if the cut condition is satisfied.*

Again, for a proof see Frank [6].
In fact, the proofs of Theorems 18, 19, 20 and 21 all yield polynomial-time algorithms for finding the required paths. These theorems also imply the following:

Theorem 22. *Let G be planar, and let* (30), (33) *or* (34) *be satisfied. Then problem* (10) *has a half-integral solution if and only if the cut condition is satisfied.*

More generally,

Theorem 23. *Let G be planar, and let* (30), (33) *or* (34) *be satisfied. Let* $d \in \mathbb{Z}_+^k$ *and* $c \in \mathbb{Z}_+^E$. *Then there exists a half-integral multicommodity flow if and only if the cut condition* (18) *is satisfied.*

Duality

Similar to the cut packing results in Section 3 dual to Theorem 9, there are theorems dual to Theorems 18, 19, 20 and 21.

The following result (Hurkens, Schrijver and Tardos [13]) is dual to the Okamura-Seymour theorem (Theorem 18):

Theorem 24. *Let G be a planar bipartite graph. Then there exist pairwise disjoint cuts* $\delta(X_1), \ldots, \delta(X_t)$ *so that for each pair of vertices u, v on the outer boundary:*

(35) $dist_G(u, v) =$ number of cuts $\delta(X_j)$ separating u, v.

In fact, this can be derived from the Okamura-Seymour theorem, as we will show now. Let

(36) $(v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k)$

be the vertices and edges on the outer boundary of G, where $v_k = v_0$. Define for each pair e_i, e_j :

(37) $r(e_i, e_j) := \frac{1}{2}(dist_G(v_{i-1}, v_{j-1}) + dist_G(v_i, v_j) - dist_G(v_{i-1}, v_j) - dist_G(v_i, v_{j-1}))$.

It is not difficult to see that this number is 0 or 1 (as G is bipartite and planar). Let Q be the set of pairs $\{e_i, e_j\}$ with $r(e_i, e_j) = 1$. Now for each v_g, v_h one has:

(38) $dist_G(v_g, v_h) =$ number of pairs $\{e_i, e_j\} \in Q$ crossing $\{v_g, v_h\}$.

Here $\{e_i, e_j\}$ *crosses* $\{v_g, v_h\}$ if v_g and v_h belong to different components of the circuit (36) after deleting e_i and e_j. Equality (38) follows from (assuming without loss of generality $0 = g < h < k$) :

(39) number of pairs $\{e_i, e_j\} \in Q$ crossing $\{v_g, v_h\} = \sum_{i=1}^{h} \sum_{j=h+1}^{k} r(e_i, e_j) = \frac{1}{2} \sum_{i=1}^{h} \sum_{j=h+1}^{k} (dist_G(v_{i-1}, v_{j-1}) + dist_G(v_i, v_j) - dist_G(v_{i-1}, v_j) - dist_G(v_i, v_{j-1}))$ $= dist_G(v_0, v_h)$

(by cancellation).

Now we can apply the Okamura-Seymour theorem to a slight modification of the dual graph of G, so that (38) implies that for each $\{e_i, e_j\} \in Q$ there exists a cut $\delta(X_{ij})$ containing e_i and e_j, in such a way that the $\delta(X_{ij})$ are pairwise disjoint. By (38) again, these cuts have the required property (35).

In [49] it is shown that the more general dual to Okamura's theorem (Theorem 19) also holds:

Theorem 25. *Let G be a planar bipartite graph, and let I_1 and I_2 be two of its faces. Then there exist pairwise disjoint cuts $\delta(X_1), \ldots, \delta(X_t)$ so that* (35) *holds for each pair of vertices u, v with $u, v \in bd(I_1)$ or $u, v \in bd(I_2)$.*

We do not see a direct way of deriving this from Okamura's theorem. Similar results hold for the duals of Theorem 19 and 20:

Theorem 26. *Let G be a planar bipartite graph, and let $r_1, s_1, \ldots, r_k, s_k$ be pairs of vertices so that* (33) *or* (34) *is satisfied. Then there exist pairwise disjoint cuts $\delta(X_1), \ldots, \delta(X_t)$ so that for each $i = 1, \ldots, k$:*

(40) $dist_G(r_i, s_i) =$ number of cuts $\delta(X_j)$ separating r_i and s_i.

With respect to (33) this follows from the results in [46]. For (34), this follows from the "sums of circuits" theorem of Seymour [51], as was communicated to me by A.V. Karzanov: Let $H = (W, F)$ be a planar graph, and let $g : F \longrightarrow \mathbb{Z}_+$ be so that $\sum_{e \ni v} g(e)$ is even for each vertex v; Seymour's theorem says that g is a nonnegative integral combination of incidence vectors of circuits in H, if and only if

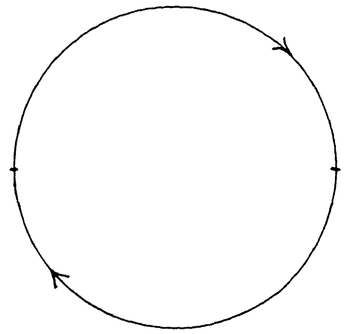
(41) $g(e') \leq \sum_{e \in D \setminus e'} g(e)$

for each cut D and each $e' \in D$. Theorem 26 is derived by applying Seymour's theorem to the graph H dual to $(V, E \cup \{\{r_1, s_1\}, \ldots, \{r_k, s_k\}\})$, with $g(e) := 1$ for edge e of H dual to an edge in E, and $g(e) := dist_G(r_i, s_i)$ for edge e of H dual to $\{r_i, s_i\}(i = 1, \ldots, k)$.

The Projective Plane and the Klein Bottle

Some of the results have an analogue in terms of compact surfaces. First consider the projective plane S. It arises from the disk

(42)



by identifying opposite points. There are two types of simple closed curves on S : the homotopically trivial closed curves, and the homotopically nontrivial closed curves (which form one homotopy class).

The homotopically trivial closed curves are those closed curves C whose removal disconnects S. The homotopically nontrivial closed curves are not disconnecting.

The homotopically trivial closed curves are also those closed curves C which are *orientation-preserving*, i.e., after one turn of C the meaning of "left" and "right" is not changed. The homotopically nontrivial closed curves are those closed curves C which are *orientation-reversing*, i.e., after one turn of C the meaning of "left" and "right" is exchanged.

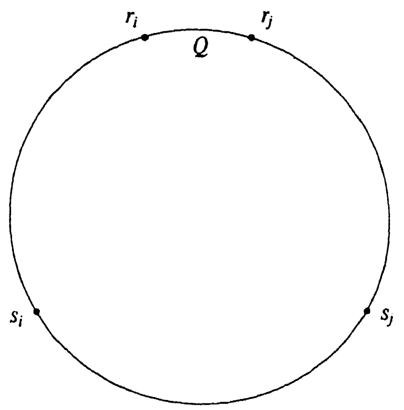
Now Lins [21] proved:

Theorem 27. *Let $G = (V, E)$ be an eulerian graph embedded on the projective plane S. Then the maximum number of pairwise edge-disjoint homotopically nontrivial circuits in G is equal to the minimum number of edges intersecting all homotopically nontrivial circuits.*

This theorem can be derived from the Okamura-Seymour theorem (Theorem 18) as follows. Let $F \subseteq E$ be a minimum set of edges intersecting all homotopically nontrivial circuits in G. It is not difficult to see that there exists a homotopically nontrivial simple closed curve D in S so that F is the set of edges intersected by D. Removing D from S gives a disk, on which $G' := (V, E \backslash F)$ is embedded. Let $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$ be the collection of pairs of end points of the edges in F (so $k = |F|$). The fact that F has minimum size implies that the cut condition (11) is satisfied with respect to $G', r_1, s_1, \ldots, r_k, s_k$. Hence by the Okamura-Seymour theorem there exist pairwise edge-disjoint paths $P_1, \ldots, P_k$ in G' connecting $r_1, s_1; \ldots; r_k, s_k$, respectively. Extending these paths with the edges in F gives a set of k circuits as required.

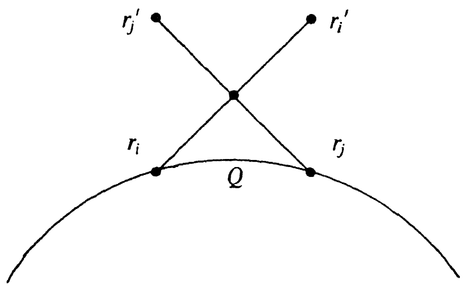
In fact, by a construction of Lins [21], one can derive conversely the Okamura-Seymour theorem from Lins' theorem (Theorem 27). Indeed, in the Okamura-Seymour theorem one may assume without loss of generality that all pairs r_i, s_i and r_j, s_j are crossing (with respect to the unbounded face). If they are not, there are two pairs r_i, s_i and r_j, s_j so that (may be after interchanging r_i and s_i) r_i, r_j, s_j, s_i are in this order on the boundary of the unbounded face (clockwise, say), and so that the path Q from r_i to r_j along this boundary (clockwise) does not contain any other vertices from $r_1, s_1, \ldots, r_k, s_k$:

(43)



Now extend G, in the unbounded face, as follows:

(44)



Replace r_i by r_i' and r_j by r_j'. It is not difficult to see that both the conditions and the conclusion of the Okamura- Seymour theorem are invariant under this modification.

After a finite number of such modifications we obtain a situation where $r_1, s_1, \ldots, r_k, s_k$ are pairwise crossing. After that we can embed the graph $(V, E \cup \{\{r_1, s_1\}, \ldots, \{r_k, s_k\}\})$ in the projective plane, in such a way that a circuit is orientation-reversing if and only if it contains an odd number of edges from $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$. If the cut condition (11) is satisfied, the minimum size of an edge set intersecting all orientation-reversing circuits is k. Hence by Lins' theorem, there exist k pairwise edge-disjoint orientation-reversing circuits, each of which cannot contain more than one edge from $\{r_1, s_1\}, \ldots, \{r_k, s_k\}$. Hence each contains exactly one such edge. It gives in the original graph G paths as required.

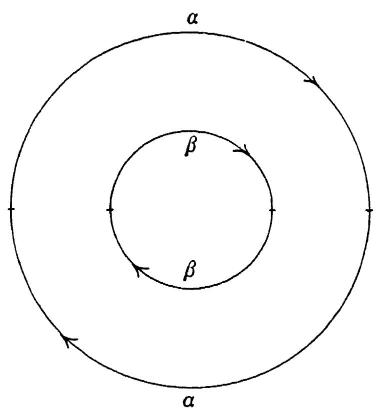
By passing over to the surface dual, Theorem 27 gives:

Theorem 28. *Let $G = (V, E)$ be a bipartite graph embedded on the projective plane S. Then the minimum length of an orientation-reversing circuit is equal to the maximum number of pairwise disjoint edge sets, each intersecting all orientation-reversing circuits.*

Theorems 27 and 28 on the projective plane, parallel to the Okamura-Seymour theorem, can be extended as follows to the Klein bottle, extending Okamura's theorem (Theorem 19) and Theorem 20.

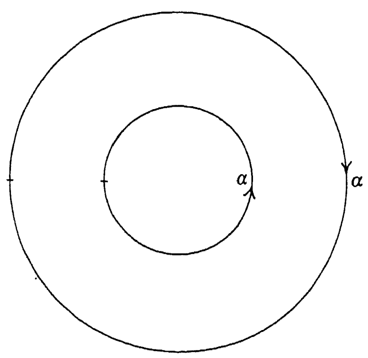
Note that the Klein bottle can be constructed from the cylinder in two possible ways. First, we can identify opposite points on one boundary, and similarly identify opposite points on the other boundary:

(45)



A second representation also comes from the cylinder. Now we identify one boundary in clockwise orientation with the other boundary in anti-clockwise orientation:

(46)



This is the usual representation of the Klein bottle.

Now in [46] we showed:

Theorem 29. *Let G be an eulerian graph embedded on the Klein bottle. Then the maximum number of pairwise edge-disjoint orientation-reversing circuits in G is equal to the minimum number of edges intersecting all orientation-reversing circuits.*

This can be derived from Theorems 19 and 20, in a similar way as Lins' theorem is derived from Theorem 18. In fact Theorems 19 and 20 correspond to the two representations of the Klein bottle described above. It is not difficult to see (by adding a "cross-cap") that Theorem 29 implies Lins' theorem.

Similarly, from Theorem 25 one can derive an extension of Theorem 28:

Theorem 30. *Let G be a bipartite graph embedded on the Klein bottle. Then the minimum length of an orientation-reversing circuit in G is equal to the maximum number of pairwise disjoint edge sets, each intersecting all orientation-reversing circuits.*

5. Vertex-Disjoint Homotopic Paths and Trees

The problem:

(47) Given: – a planar graph $G = (V, E)$,
– pairs $r_1, s_1, \ldots, r_k, s_k$ of vertices,
find: – pairwise vertex-disjoint paths $P_1, \ldots, P_k$ in G, where P_i connects r_i and s_i (for $i = 1, \ldots, k$),

is NP-complete. So in order to solve this problem, one seemingly is bound to nonpolynomial or suboptimal methods, like enumeration and heuristics.

Pinter [31] proposed to make use of the topology of the plane, and to classify the possible solutions after their homotopy with respect to certain "holes" in the plane.

That is, select a number of faces $I_1, \ldots, I_p$ (including the unbounded face), such that $r_1, s_1, \ldots, r_k, s_k$ all are on the boundary of $I_1 \cup \ldots \cup I_p$. Two curves C, C' : $[0,1] \longrightarrow \mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ are called *homotopic* (in the space $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$) if there exists a continuous function $\Phi : [0,1] \times [0,1] \longrightarrow \mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ such that

(48)
$$\Phi(x,0) = C(x), \Phi(x,1) = C'(x),$$
$$\Phi(0,x) = C(0), \Phi(1,x) = C(1)$$

for all $x \in [0,1]$. Note that this implies that $C(0) = C'(0)$ and $C(1) = C'(1)$.

So C and C' are homotopic if C can be shifted continuously over $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ to C', without changing the beginning point or the end point of the curve.

Homotopy determines an equivalence relation between curves. Curves C, C' beeing homotopic is denoted by $C \sim C'$.

Since each path in G can be considered as a curve in $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$, it also belongs to some homotopy class. So one approach to solve problem (48) is first to choose for each pair r_i, s_i a homotopy class of curves connecting r_i and s_i (represented by one curve C_i), and next to find paths $P_1, \dots, P_k$ so that $P_i \sim C_i$ (for $i = 1, \dots, k$).

This approach can be done in an enumerative way, by enumerating all possible choices of homotopy classes (there are some direct ways of ensuring finiteness of this enumeration, by excluding trivially infeasible choices), or alternatively in a heuristic way, by guessing a choice of homotopy classes, and locally improving it in case it turns out infeasible.

This approach asks for solving the following problem:

(49) Given:
- a planar graph $G = (V, E)$, embedded in $\mathbb{R}^2$,
- faces $I_1, \dots, I_p$ of G (including the unbounded face),
- curves $C_1, \dots, C_k$ with end points on $bd(I_1 \cup \dots \cup I_p)$,

find:
- pairwise vertex-disjoint simple paths $P_1, \dots, P_k$ where P_i is homotopic to C_i in $\mathbb{R}^2 \setminus (I_1 \cup \dots \cup I_p)$ (for $i = 1, \dots, k$).

The following was shown in [7]:

Theorem 31. *Problem* (49) *is solvable in polynomial time.*

The proof in [7] used the ellipsoid method. Below we shall give a sketch of the method based on [42]. In [48] we give an algorithm with running time $0(|V|^2 \cdot log^2 |V|)$. Earlier, a polynomial-time algorithm for (49) was given by Leiserson and Maley [20] in case G is a "grid" graph - an important case for VLSI-design. Moreover, Robertson and Seymour [33] gave a polynomial-time algorithm for (49) if $p = 1$ (which is Theorem 2) and if $p = 2$.

In fact, in [44] we gave an polynomial-time algorithm for a problem more general than (49), viz. where one wants to connect sets of points by trees instead of paths - see below.

Sketch of the Algorithm for (49)

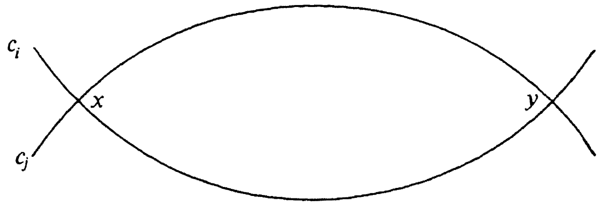
We give a sketch of the algorithm of [42] for problem (49), leaving out many details. It consists of four basic steps:

(50)
I. Uncrossing $C_1, \dots, C_k$,
II. Determining the system $Ax \leq b$,
III. Solving the system $Ax \leq b$ in integers,
IV. Shifting the curves.

I. Uncrossing $C_1, \dots, C_k$

We first "uncross" $C_1, \dots, C_k$ so as to make them simple and pairwise disjoint. That is, if C_i and C_j have a crossing x, they should have a second crossing y so that the parts of C_i and C_j in between of x and y are homotopic:

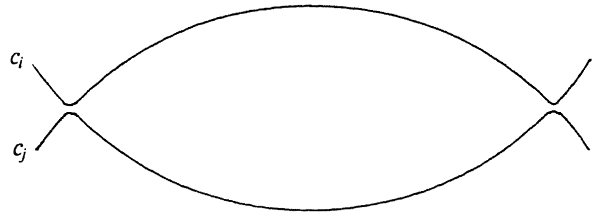
(51)



So roughly speaking, none of the faces $I_1, \ldots, I_p$ is contained in the region enclosed. If C_i and C_j have a crossing x, and they would not have a second crossing y with this property, then problem (49) has no solution.

Now replace (51) by:

(52)



Now the new C_i and C_j are homotopic to the original C_i and C_j. In a similar way we can uncross self-crossings of any C_i. Repeating this we will end up with

(53) curves $\tilde{C}_1, \ldots, \tilde{C}_k$ in $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ being simple and pairwise disjoint, so that $\tilde{C}_i \sim C_i$ for $i = 1, \ldots, k$

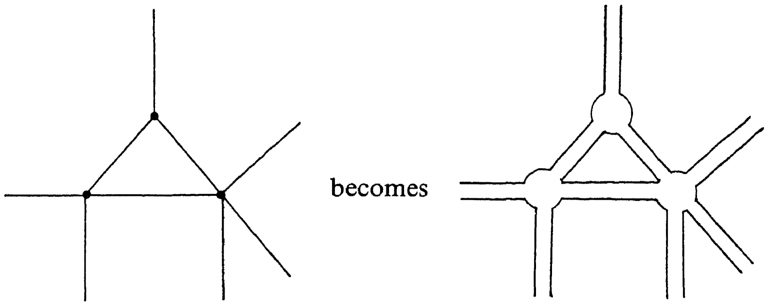
(or curves with this property do not exist at all, in which case (49) trivially has no solution). Without loss of generality, $\tilde{C}_i = C_i$ for all i.

II. Determining the System $Ax \leq b$

We next determine a system $Ax \leq b$ of linear inequalities (where A is a matrix and b is a column vector).

First "blow up" the graph G slightly. That is, each vertex v of G becomes a disk D_v, and each edge e becomes a "channel":

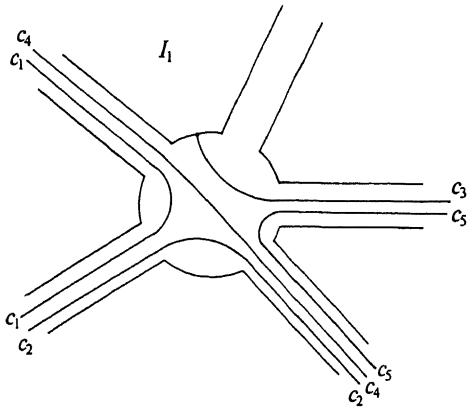
(54)



Let H be the blown-up "graph". Each face F of G corresponds naturally to a "face" F' of H. We may assume (by shifting slightly), that $I_1' = I_1, \ldots, I_p' = I_p$.

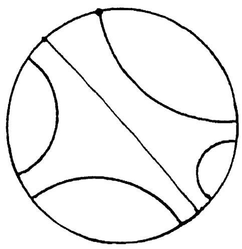
We can "push" the C_i so that they are simple and pairwise disjoint and they are in the interior of H. So we get, e.g.,

(55)



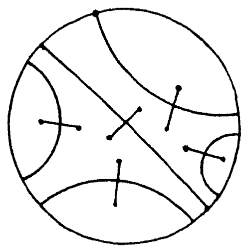
Consider now any disk D_v together with all C_i passing D_v :

(56)



Each time a curve C_i passes D_v, we introduce a small line segment crossing C_i in D_v :

(57)



We do this for each vertex v of G. This gives us a set $\mathscr{L}$ of pairwise disjoint line segments. Let U be the set of end points of line segments in $\mathscr{L}$. So $|U| = 2|\mathscr{L}|$. We call $u, u' \in U$ *mates* if they are the two end points of one line segment in $\mathscr{L}$.

Next introduce a variable x_u for each $u \in U$. The value of this variable x_u will stand for the distance over which we will shift the corresponding curve C_i to obtain a path P_i in G as required.

We give four classes of linear inequalities in the $x_u (u \in U)$. First:

(58) $$x_u + x_{u'} = 0 \quad \text{if } u \text{ and } u' \text{ are mates.}$$

Second consider $u, u' \in U$ so that u is end point of a line segment crossing C_i and u' is end point of a line segment crossing C_j with $j \neq i$, so that u and u' belong to the same component of:

(59) $$\mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p \cup C_1[0,1] \cup \ldots \cup C_k[0,1]).$$

Let for any curve D in $\mathbb{R}^2$:

(60) $$\varphi(D) := \text{the number of faces } F' \text{ of } H \text{ passed by } D \text{ (counting multiplicities).}$$

Define:

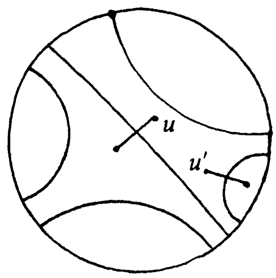
(61) $$\beta_{u,u'} := min\,\{\varphi(D) \mid D \text{ is homotopic in } \mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p) \text{ to some curve in (59) connecting } u \text{ and } u'\}.$$

Now we require:

(62) $$x_u + x_{u'} \leq \beta_{u,u'} - 1.$$

It means that if curve C_i is shifted at u over a distance x_u (in the direction of u), then curve C_j should be shifted over a distance of at least $x_u + 1 - \beta_{u,u'}$, in the negative direction. In particular, (62) gives that if u and u' belong to the same disk D_v as in:

(63)



then $x_u + x_{u'} \leq -1$.

Third, consider $u, u' \in U$ so that u is end point of a line segment in $\mathscr{L}$ crossing C_i, and u' is end point of a line segment in $\mathscr{L}$ also crossing C_i. Moreover, let there exist a curve D satisfying:

(64) (i) D is a curve in (59) connecting u and u',

(ii) D is not homotopic to any curve in $C_i[0,1] \cup \bigcup_{\ell \in \mathscr{L}} \ell$ connecting u and u'.

Now let

(65) $$\beta_{u,u'} := \min\{\varphi(\tilde{D}) \mid \tilde{D} \text{ is homotopic to some curve } D \text{ satisfying (64)}\}.$$

We require:

(66) $$x_u + x_{u'} \leq \beta_{u,u'} - 1.$$

This includes the case $u = u'$, where (64) (ii) means that D is homotopically nontrivial, and where (66) becomes $2x_u \leq \beta_{u,u} - 1$.

Finally, for any $u \in U$ let

(67) $$\beta_u := min\{\varphi(D) \mid D \text{ is homotopic in } \mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p) \text{ to some curve in (59) connecting } u \text{ and } bd(I_1 \cup \ldots \cup I_p)\}.$$

We require:

(68) $$x_u \leq \beta_u.$$

It means that we should not shift curve C_i over one of the "holes" $I_1, \ldots, I_p$.

By $Ax \leq b$ we denote the system of linear inequalities made up by (58), (62), (66) and (68). It can be shown that the right hand sides in these inequalities can be calculated in polynomial time.

III. Solving the System $Ax \leq b$ in Integers

In general, solving a system of linear inequalities in integers is NP-complete. However, our matrix A is of a special type. It satisfies:

(69) $$\sum_{j=1}^{n} |a_{ij}| \leq 2 \qquad \text{for } i = 1, \ldots, m,$$

where $A = (a_{ij})$ has order $m \times n$, say. In that case, $Ax \leq b$ can be solved in integers, e.g., with the "Fourier-Motzkin elimination method" (cf. [39]).

This method eliminates variables one by one. Order the inequalities in $Ax \leq b$ as:

(70) $$\begin{aligned} 2x_1 &\leq \gamma_1 \\ -2x_1 &\leq \gamma_2 \\ x_1 + a_1 x' &\leq \delta_1 \\ \vdots \quad & \vdots \;\; \vdots \\ x_1 + a_{m'} x' &\leq \delta_{m'} \end{aligned}$$

$$\begin{aligned} -x_1 + a_{m'+1}x' &\leq \delta_{m'+1} \\ \vdots \quad & \vdots \; \vdots \\ -x_1 + a_{m''}x' &\leq \delta_{m''} \\ a_{m''+1}x' &\leq \delta_{m''+1} \\ &\vdots \; \vdots \\ a_m x' &\leq \delta_m \end{aligned}$$

where $a_1, \ldots, a_m$ are row vectors of dimension $n-1$, where $x' := (x_2, \ldots, x_n)^T$, taking possibly $\gamma_1 = \infty$ or $\gamma_2 = \infty$.

Now if $\gamma_1 + \gamma_2 < 0$ then (70) clearly has no solution. If $\gamma_1 + \gamma_2 = 0$ and $\gamma_1 = -\gamma_2$ is odd, then (70) has no integer solution. So we may assume:

(71) $$\lceil -\frac{1}{2}\gamma_2 \rceil \leq \lfloor \frac{1}{2}\gamma_1 \rfloor.$$

We can put (70) in another form:

(72) $$\begin{aligned} -\frac{1}{2}\gamma_2 \leq x_1 \leq \frac{1}{2}\gamma_1, & \\ a_j x' - \delta_j \leq x_1 \leq \delta_i - a_i x' \quad & i = 1, \ldots, m'; j = m'+1, \ldots, m''; \\ a_i x' \leq \delta_i \quad & i = m''+1, \ldots, m. \end{aligned}$$

Eliminating x_1 gives:

(73) $$\begin{aligned} (a_i + a_j)x' \leq \delta_i + \delta_j \quad & i = 1, \ldots, m'; j = m'+1, \ldots, m''; \\ a_i x' \leq \frac{1}{2}\gamma_2 + \delta_i \quad & i = 1, \ldots, m'; \\ a_j x' \leq \frac{1}{2}\gamma_1 + \delta_j \quad & j = m'+1, \ldots, m''; \\ a_i x' \leq \delta_i \quad & i = m''+1, \ldots, m. \end{aligned}$$

If (70) has an integer solution, also (73) must have an integer solution. System (73) is again of the same type as the original system; i.e., the corresponding matrix satiesfies (69) again. So we can solve (73) recursively. Let x' be an integral solution to (73). Hence, using (71), we have:

(74) $$\lceil \max\{\frac{-1}{2}\gamma_2, \max_{m'+1 \leq j \leq m''}(a_j x' - \delta_j)\} \rceil \leq \lfloor \min\{\frac{1}{2}\gamma_1, \min_{1 \leq i \leq m'}(\delta_i - a_i x')\} \rfloor.$$

This implies that we can find an *integer* x_1 satisfying (72). Thus we have found an integer solution to (70).

The polynomial running time bound of this method follows from the fact that the system (73) can be reduced to a system with $O(n^2)$ inequalities: for any set of inequalities with equal left hand side, we consider only that one with lowest right hand side.

IV. Shifting the Curves

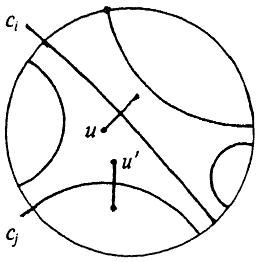
We call the integers x_u found by solving $Ax \leq b$ the *shift numbers*. They determine the distance and direction of shifting of the curves C_i. We carry out this shifting in small steps. Roughly it works as follows.

If all x_u are equal to 0, then no two distinct curves C_i pass the same disk D_v. Indeed, if two different curves C_i and C_j would pass disk D_v, then there are two different curves C_i and C_j passing D_v in such a way that they are incident to the same component of

$$D_v \setminus (C_1[0,1] \cup \ldots \cup C_k[0,1]), \tag{75}$$

like in:

(76)



Let u and u' be as indicated. By (62), $x_u + x_{u'} \leq -1$, contradicting $x_u = x_{u'} = 0$.

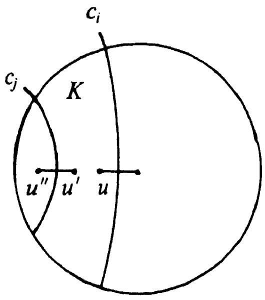
Moreover, if one curve C_i passes a disk D_v more than once, we can similarly derive from (66) that the "loop" in between of the two passes of C_i through D_v is homotopic to some curve in D_v. So we can shortcut C_i. Repeating this, we obtain $C_1, \ldots, C_k$ so that each D_v is passed at most once in total. Shrinking H to G the curves $C_1, \ldots, C_k$ transform to pairwise disjoint simple paths in G as required.

If not all x_u are 0, select one with $x_u = M > 0$ as large as possible. Let u belong to component K of

$$D_v \setminus (C_1[0,1] \cup \ldots \cup C_k[0,1]). \tag{77}$$

Suppose there is another point $u' \in U$ in K:

(78)



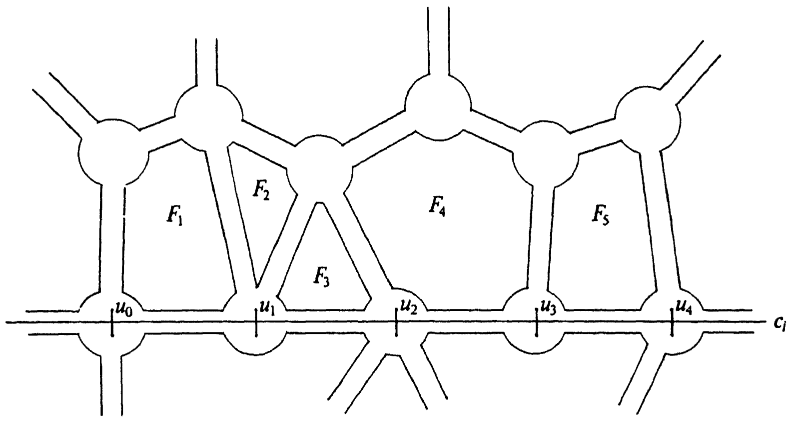
If $j \neq i$, then by (62), $x_{u'} + x_u \leq -1$, and hence $x_{u''} = -x_{u'} \geq x_u + 1 = M + 1$, contradicting the maximality of x_u. If $j = i$, then the loop in between of two passes of C_i through D_v is homotopic to some curve in D_v, so we can shortcut C_i.

So we may assume that no such u' exists. It means that K is "on the border" of D_v. Consider a longest subcurve of C_i so that a consecutive series of line segments has end point u with $x_u = M$:

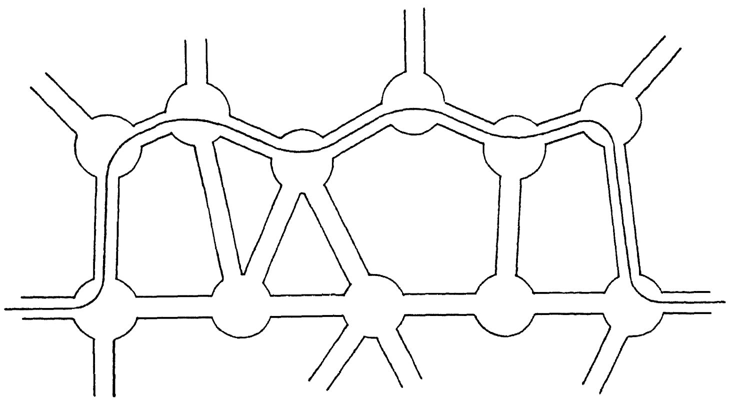
(79)


with $x_{u_0} < M, x_{u_1} = x_{u_2} = \ldots = x_{u_{s-1}} = x_{u_s} = M$ and $x_{u_{s+1}} < M$. (Such a longest path exists, as at the beginning and end of C_i we have $x_u = 0$ (by (58) and (68)).)

Consider a neighbourhood of H at the same side of C_i as $u_1, \ldots, u_s$:

(80)


By (68), none of the faces $F_1, \ldots, F_5$ (in this example) belongs to $I_1, \ldots, I_p$. So we can shift C_i as:

(81)


We introduce new line segments $\ell_1, \ldots, \ell_s$, crossing the new part of C_i in the corresponding disks. Let $u'_1, \ldots, u'_{s'}$ be the end points at the lower side in (81). So we replace $u_1, \ldots, u_s$ and their mates by $u'_1, \ldots, u'_{s'}, u''_{1'}, \ldots, u''_{s'}$. The same we do for the variables. The new variables we set:

(82)
$$x_{u'_1} := x_{u'_2} := \ldots := x_{u'_{s'}} := M - 1,$$
$$x_{u''_1} := x_{u''_2} := \ldots := x_{u''_{s'}} := -M + 1,$$

leaving the remaining variables invariant. It is not difficult to see that (generally) we obtain in this way an integral solution for the system of linear inequalities corresponding to the modified system.

Repeating this "local" shifting, we obtain after a polynomial number of steps a system with all x_u equal to 0, in which case the C_i give paths in G as required.

On the Correctness of the Method

The correctness of the method follows from the following fact:

(83) problem (49) has a solution $\Leftrightarrow$ system $A \le b$ has an integral solution.

The implication $\Leftarrow$ is proved by showing the correctness of the above shifting process. The implication $\Rightarrow$ is proved by deriving shift numbers x_u from any solution of (49).

The implication $\Rightarrow$ can also derived in the following way. Let $A = (a_{ij})$ be any integral $m \times n$-matrix satisfying

(84)
$$\sum_{j=1}^{n} |a_{ij}| \le 2 \quad \text{for each} \quad i = 1, \ldots, m.$$

Let b be an integral column vector of dimension m. In characterizing the solvability of $Ax \le b$ in integers, consider first the case that each row of A contains one $+1$ and one -1. Then A is the incidence matrix of some directed graph. We can consider b as a length function on the edges of this directed graph. Then a solution x of $Ax \le b$ is called a *potential*. It satisfies:

(85)
$$x_w - x_v \le b_{vw} \quad \text{for any edge} \quad vw.$$

As is well-known, such an integral potential exists if and only if each directed cycle has nonnegative length.

The general case can be studied in terms of *bidirected graphs*. We can in fact identify the matrix A with a bidirected graph. The vertices are identified with the columns (or column indices) of A, and the edges with the rows (or row indices) of A. An edge connects v and w if $a_{ev} \neq 0$ and $a_{ew} \neq 0$. So we have $++$ edges, $+-$ edges, and $-$ edges, indicated as

(86) 

A row with a ± 2 can be seen as a loop. There are two types: ++ loops and – loops, indicated as:

(87) 

We call a row with only one ± 1 an *end*, at the corresponding vertex v. They can be indicated as:

(88) 

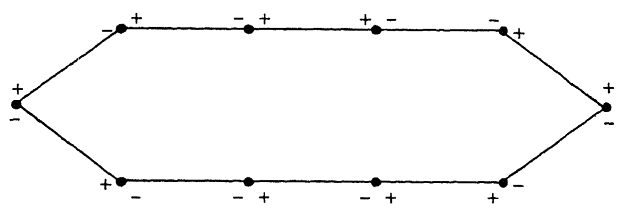
Call a sequence

(89) $$(v_0, e_1, v_1, \dots, e_d, v_d)$$

a *bidirected cycle* if:

(90) (i) $v_0 = v_d$;
(ii) e_i is an edge or loop connecting v_{i-1} and v_i $(i = 1, \dots, d)$;
(iii) $a_{e_iv_i} \cdot a_{e_{i+1}v_i} < 0$ (for $i = 1, \dots, d-1$), and $a_{e_1v_0} \cdot a_{e_dv_0} < 0$

(the vertices $v_1, \dots, v_d$ need not all be distinct). An example is:

(91) 

A first necessary condition for the existence of a solution of $Ax \le b$ is:

(92) each bidirected cycle has nonnegative length

(where the *length* of cycle (89) is $\sum_{j=1}^{d} b_{e_j}$). This follows from:

(93) $$\sum_{j=1}^{d} b_{e_j} \ge \sum_{j=1}^{d} (a_{e_jv_{j-1}}x_{v_{j-1}} + a_{e_jv_j}x_{v_j}) = \sum_{j=1}^{d} (a_{e_jv_j} + a_{e_{j+1}v_j})x_{v_j} = 0$$

(taking $e_{d+1} := e_1$, and assuming for simplicity that no e_j is a loop - the general case is left as an exercise).

Call a sequence

(94) $$(e_1, v_1, e_2, v_2, \ldots, v_{d-2}, e_{d-1}, v_{d-1}, e_d)$$

a *link* if

(95) (i) e_1 is an end at v_1 and e_d is an end at v_{d-1};

(ii) e_i is an edge or loop connecting v_{i-1} and $v_i (i = 2, \ldots, d-1)$;

(iii) $a_{e_i v_i} \cdot a_{e_{i+1} v_i} < 0$ (for $i = 1, \ldots, d-1$).

As a second necessary condition for the existence of a solution of $Ax \leq b$ we have:

(96) each link has nonnegative length.

It can be shown that (92) and (96) together are sufficient for the existence of a rational solution of $Ax \leq b$. However, for an integral solution we need one further condition. Call a cycle (89) *doubly-odd* if there exists a t with $0 < t < d$ so that:

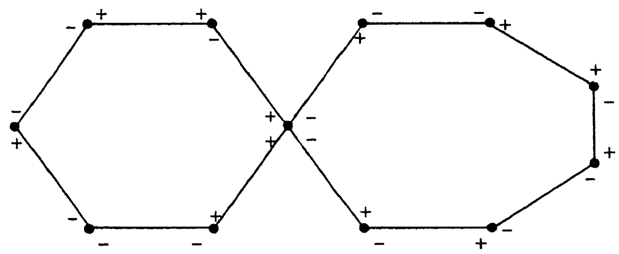
(97) (i) $v_0 = v_t = v_d$;

(ii) $a_{e_1 v_0} \cdot a_{e_t v_t} > 0$ and $a_{e_{t+1} v_t} \cdot a_{e_d v_d} > 0$;

(iii) $\sum_{j=1}^{t} b_{e_j}$ is odd and $\sum_{j=t+1}^{d} b_{e_j}$ is odd.

An example of a cycle satisfying (i) and (ii) is:

(98)



Now a necessary condition for the existence of an integral solution of $Ax \leq b$ is:

(99) each doubly-odd cycle has positive length.

This follows from (assuming again for simplicity that no e_j is a loop):

(100)
$$\sum_{j=1}^{t} b_{e_j} \geq \sum_{j=1}^{t}(a_{e_j v_{j-1}} x_{v_{j-1}} + a_{e_j v_j} x_{v_j}) =$$
$$= a_{e_1 v_0} v_{v_0} + \sum_{j=1}^{t-1}(a_{e_j v_j} + a_{e_{j+1} v_j}) x_{v_j} + a_{e_t v_t} x_{v_t} = 2x_{v_0}.$$

Since the left hand side is odd, we should have strict inequality if x is integral. Hence we have strict inequality in (93).

Now conditions (92), (96) and (99) are sufficient for the existence of an integral solution of $Ax \leq b$:

Theorem 32. *Let A be an integral matrix satisfying* (84), *and let b be an integral column vector. Then $Ax \leq b$ has an integral solution x, if and only if:*

(101) (i) each bidirected cycle has nonnegative length;
(ii) each link has nonnegative length;
(iii) each doubly-odd cycle has positive length.

It is not difficult to derive a proof of this theorem with the help of the Fourier-Motzkin elimination method described above.

From Theorem 32 one can derive the following theorem [42,43]:

Theorem 33. *Problem* (49) *has a solution if and only if:*

(102) (i) there exist pairwise disjoint simple curves $C'_1, \ldots, C'_k$ in $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ so that $C'_i \sim C_i$ (for $i = 1, \ldots, k$);
(ii) for each curve $D : [0,1] \longrightarrow \mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ with end points on $bd\ (I_1 \cup \ldots \cup I_p)$ one has:

$$cr\ (G, D) \geq \sum_{i=1}^{k} mincr\ (C_i, D);$$

(iii) for each doubly-odd closed curve

$$D : S_1 \longrightarrow \mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$$

not passing obligatory points one has:

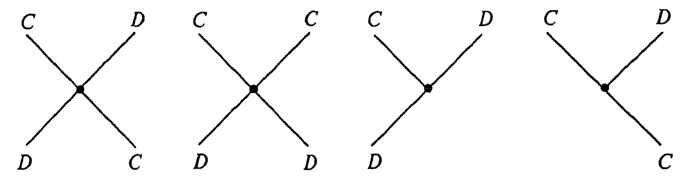
$$cr\ (G, D) > \sum_{i=1}^{k} mincr\ (C_i, D).$$

Here we use the following notation and terminology. We denote:

(103)
$$cr\ (G, D) := |\{x \in [0,1] \mid D(x) \in G\}|,$$
$$cr\ (C, D) := |\{(x, y) \in [0,1] \times [0,1] \mid C(x) = D(y)\}|;$$
$$mincr\ (C, D) := min\ \{cr(\tilde{C}, \tilde{D}) \mid \tilde{C} \sim C, \tilde{D} \sim D\}.$$

So $cr\ (C, D)$ counts the number of intersections of C and D, which can be of several types:

(104)



(crossing) (touching)

By S_1 we denote the unit circle in the complex plane $\mathbb{C}$. A closed curve $D : S_1 \longrightarrow \mathbb{R}^2$ is called *doubly-odd* if it is the concatenation of two closed curves $D_1, D_2 : S_1 \longrightarrow \mathbb{R}^2$, with $D_1(1) = D_2(1) \notin G$, so that

(105)
$$cr\ (G, D_1) + \sum_{i=1}^{k} kr\ (C_i, D_1) \text{ is odd, and}$$
$$cr\ (G, D_2) + \sum_{i=1}^{k} kr\ (C_i, D_2) \text{ is odd.}$$

Here $kr\ (C, D)$ denotes the number of crossings of C and D (cf. (104)).

An *obligatory point* is a point $p \in \mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$ so that, for some $i = 1, \ldots, k$, each C_i' homotopic to C_i passes p.

Two closed curves $D, D' : S_1 \longrightarrow \mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$ are called *homotopic* (or *freely homotopic*) denoted by $D \sim D'$, if there exists a continuous function $\Phi : S_1 \times [0,1] \longrightarrow \mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$ so that

(106)
$$\Phi(z, 0) = D(z) \text{ and } \Phi(z, 1) = D'(z)$$

for all $z \in S_1$ (so no base point is fixed). Again we denote:

(107)
$$\begin{aligned} cr\ (G, D) &:= |\{z \in S_1 \mid D(z) \in G\}|, \\ cr\ (C, D) &:= |\{(y, z) \in [0,1] \times S_1 \mid C(y) = D(z)\}|, \\ mincr\ (C, D) &:= min\ \{cr\ (\tilde{C}, \tilde{D}) \mid \tilde{C} \sim C, \tilde{D} \sim D\}. \end{aligned}$$

Theorem 33 extends a theorem of Cole and Siegel [4] for grid graphs, and a theorem of Robertson and Seymour [33] for the case $p = 2$ (i.e., one proper hole). In these two cases we can delete condition (102) (iii).

To sketch the proof of Theorem 33, we note first that it is not difficult to see that the conditions (102) are necessary. To see sufficiency, observe that we may assume that $C_1', \ldots, C_k'$ in (102) (i) are in fact equal to $C_1, \ldots, C_k$, respectively, and that they are in the "blown up" graph H as above. Construct the system $Ax \leq b$ from this. Now each inequality

(108)
$$x_u + x_{u'} \leq \beta_{u,u'} - 1$$

(62) and (66) comes from a curve D in $\mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$ connecting u and u' th $\varphi(D) = \beta_{u,u'}$. Similarly, the inequalities

09)
$$x_u \leq \beta_u$$

in (68) come from a curve D in $\mathbb{R}^2\backslash(I_1 \cup \ldots \cup I_p)$ connecting u and the boundary of some face $I_1, \ldots, I_p$ with $\varphi(D) = \beta_u$. The inequalities

(110) $$x_u + x_{u'} = 0$$

in (58) correspond to a line segment in $\mathscr{L}$ with end points u and u'. This implies that each link (94) in A corresponds to a curve D in $\mathbb{R}^2\backslash(I_1 \cup \ldots \cup I_p)$ connecting two points on $bd(I_1 \cup \ldots \cup I_p)$. Note that

(111) the number of inequalities in link (94) corresponding to a line segment in $\mathscr{L}$ is equal to $\frac{1}{2}(d-1)$.

Moreover, the length of the link is:

(112) $$\sum_{j=1}^{d} b_{e_j} = \varphi(D) - \frac{1}{2}(d-1).$$

It is not difficult to show further:

(113) $$\frac{1}{2}(d-1) = \sum_{i=1}^{k} mincr\ (C_i, D).$$

Hence condition (102) (ii) implies condition (101) (ii). It is also not difficult to see that condition (102) (ii) implies

(114) $$cr\ (G, D) \geq \sum_{i=1}^{k} mincr\ (C_i, D)$$

for each *closed* curve D in $\mathbb{R}^2\backslash(I_1 \cup \ldots \cup I_p)$: take any curve $E : [0,1] \longrightarrow \mathbb{R}^2\backslash(I_1 \cup \ldots \cup I_p)$ with $E(0) \in bd(I_1 \cup \ldots \cup I_p)$ and $E(1) = D(1)$. Then the curve

(115) $$E \cdot D^t \cdot E^{-1}$$

(for $t \in \mathbb{N}$) satisfies:

(116) $$cr\ (G, E \cdot D^t \cdot E^{-1}) = 2 \cdot cr\ (G, E) + t \cdot cr\ (G, D)$$

(assuming without loss of generality that $D(1) \notin G$). One can show that there exists a number S so that for each $t \geq 0$ and each $i = 1, \ldots, k$:

(117) $$mincr\ (C_i, E \cdot D^t \cdot E^{-1}) \geq t \cdot mincr\ (C_i, D) - S.$$

By (102) (ii) (applied to curve (115)) and (116), for each $t \geq 0$:

(118) $$t \cdot cr\ (G, D) \geq t \cdot \sum_{i=1}^{k} mincr\ (C_i, D) - kS - 2cr(G, E).$$

Hence (114) follows. In a similar way as above one can derive from (114) that (101) (i) holds. Moreover, condition (102) (iii) can be seen to imply condition (101) (iii).

Fixed Number of Holes

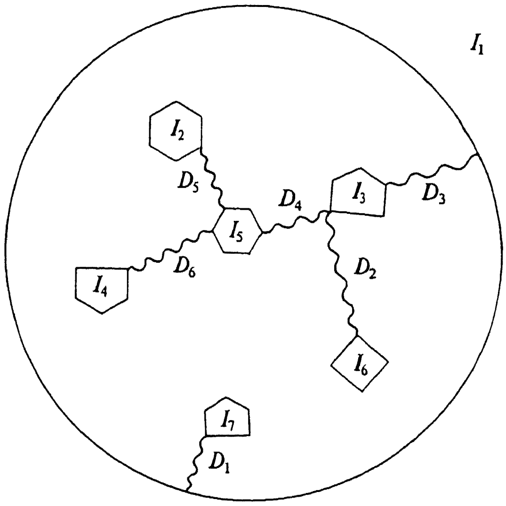
The following can be derived from Theorem 31:

Theorem 34. *For each fixed p there exists a polynomial-time algorithm for problem* (3), *whenever G is planar so that* $r_1, s_1, \ldots, r_k, s_k$ *can be covered by the boundaries of at most p faces.*

The idea of the proof is as follows. Let $r_1, s_1, \ldots, r_k, s_k$ be covered by the boundaries of faces $I_1, \ldots, I_p$ (including the unbounded face, without loss of generality). Consider $I_1, \ldots, I_p$ as holes. Now we can enumerate all "possibly feasible" homotopy classes of curves $C_1, \ldots, C_k$ (where C_i connects r_i and $s_i (i = 1, \ldots, k)$) in polynomial time.

Indeed, we only have to consider those curves which are pairwise disjoint and simple. Moreover, we can find curves $D_1, \ldots, D_{p-1}$, each connecting the boundaries of two of the faces $I_1, \ldots, I_p$, so that they form a "spanning tree" on $I_1, \ldots, I_p$. E.g.,

(119)



Note that the space obtained by deleting all holes $I_1, \ldots, I_p$ and the images of all curves $D_1, \ldots, D_{p-1}$ is simply connected.

We can take $D_1, \ldots, D_{p-1}$ so that $cr\ (G, D_j) \leq |V|$ for all $j = 1, \ldots, p-1$. Then we only have to consider those choices for the curves $C_1, \ldots, C_k$ for which

(120) $$\sum_{i=1}^{k} mincr\ (C_i, D_j) \leq |V| \quad \text{for } j = 1, \ldots, p-1,$$

since other choices obviously are infeasible. It can be shown that there are at most $|V|^p$ such choices (up to homotopy). Hence we can restrict the enumeration to a polynomial number of choices.

Theorem 34 extends Robertson and Seymour's theorem (Theorem 1) for the case that G is planar: if k is fixed, we can cover $r_1, s_1, \ldots, r_k, s_k$ by a fixed number of faces, namely at most $2k$.

Surfaces

The following theorem from [45] can be proved in a way similar to the proof of Theorem 33 above.

Theorem 35. *Let $G = (V, E)$ be a graph, embedded on a compact surface S, and let $C_1, \ldots, C_k$ be closed curves on S, each not null-homotopic. Then there exist pairwise disjoint simple closed curves $\tilde{C}_1, \ldots, \tilde{C}_k$ in G so that $\tilde{C}_i$ is homotopic to C_i for $i = 1, \ldots, k$, if and only if:*

(121) (i) there exist pairwise disjoint simple closed curves $\tilde{C}_1, \ldots, \tilde{C}_k$ on S so that $\tilde{C}_i$ is homotopic to C_i for $i = 1, \ldots, k$;

(ii) for each closed curve $D : S_1 \longrightarrow S$:

$$cr\ (G, D) \geq \sum_{i=1}^{k} mincr\ (C_i, D);$$

(iii) for each doubly-odd closed curve $D = D_1 \cdot D_2 : S_1 \longrightarrow S$ with $D_1(1) = D_2(1) \notin G$:

$$cr\ (G, D) > \sum_{i=1}^{k} mincr\ (C_i, D).$$

Here we use similar terminology as above. Thus a closed curve (on S) is a continuous function $C : S_1 \longrightarrow S$, where S_1 denotes the unit circle in the complex plane $\mathbb{C}$. It is *simple* if it is one-to-one. Two closed curves are *disjoint* if their images are disjoint.

Two closed curves C and $\tilde{C}$ are *(freely) homotopic (on S)*, in notation $C \sim \tilde{C}$, if there exists a continuous function $\Phi : S_1 \times [0,1] \longrightarrow S$ so that $\Phi(z, 0) = C(z)$ and $\Phi(z, 1) = C(z)$ for all $z \in S_1$.

Again, we call a closed curve $D : S_1 \longrightarrow S$ *doubly-odd* (with respect to $G, C_1, \ldots, C_k$) if $D = D_1 \cdot D_2$ for some closed curves D_1, D_2 satisfying:

$$(122) \qquad cr\ (G, D_1) \not\equiv \sum_{i=1}^{k} cr\ (C_i, D_1) \pmod 2,$$

$$cr\ (G, D_2) \not\equiv \sum_{i=1}^{k} cr\ (C_i, D_2) \pmod 2.$$

It is easy to see that the conditions (121) are necessary conditions. The essence of the theorem is sufficiency of (121).

Homotopic Trees

We can extend the polynomial-time algorithm for problem (49) to the following problem:

(123) Given: – a planar graph G embedded in $\mathbb{R}^2$;
– faces $I_1, \ldots, I_p$ of G (including the unbounded face);
– pairwise disjoint sets $W_1, \ldots, W_k$ of vertices of G on the boundary of $I_1 \cup \ldots \cup I_p$;
– trees $T_1, \ldots, T_k$ embedded in $\mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$, so that $W_i \subseteq V(T_i)$ for $i = 1, \ldots, k$;
find: – pairwise disjoint subtrees $\tilde{T}_1, \ldots, \tilde{T}_k$ of G so that for each $i = 1, \ldots, k$: $\tilde{T}_i$ is homotopic to T_i in $\mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$ fixing W_i.

Here two trees T and $\tilde{T}$ embedded *in* $\mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$ are called *homotopic* (in notation: $T \sim \tilde{T}$) in $\mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$ *fixing* W if:

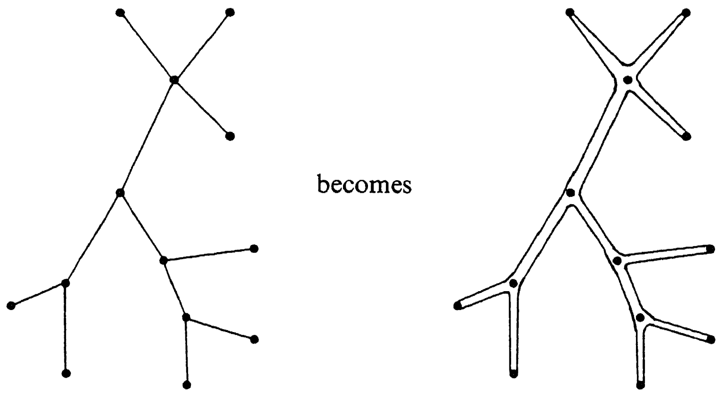
(124) (i) W is a subset both of $V(T)$ and of $V(\tilde{T})$:
(ii) for every pair of elements $w, w' \in W$, the unique simple curve in T connecting w and w' is homotopic to the unique simple curve in $\tilde{T}$ connecting w and w' (in $\mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$).

In [44] we showed:

Theorem 36. *There exists a polynomial-time algorithm for problem* (123).

The idea of the algorithm is as follows. Again we blow up the graph G slightly, as in (54), to obtain H. We replace each tree T_i by $t_i := |W_i|$ paths, following the contours of T_i. E.g.,

(125)



(assuming the end nodes are the elements of W_i). So T_i gives t_i paths $C_1, \ldots, C_{t_i}$, so that the concatenation

(126) $$K_i := C_1 \cdot C_2 \cdot \ldots \cdot C_{t_i}$$

is a simple closed curve, containing no face $I_1, \ldots, I_p$ in its interior. Let L_i denote this interior. Assuming the original $T_1, \ldots, T_k$ to be pairwise disjoint, the closed curves $K_1, \ldots, K_k$ are pairwise disjoint. We may assume that they are part of H.

Again we introduce line segments each time a curve K_i passes any disk D_v (cf. (57)). Let $\mathscr{L}$ be the set of these line segments, and let U be the set of end points of line segments in $\mathscr{L}$.

Now if u and u' are end points of one line segment in $\mathscr{L}$, crossing C_{ij} say, then one of the end points is in L_i and the other not. Call the first one the *inner* end point, and the other one the *outer* end point. Let U' be the set of inner end points, and let U'' be the set of outer end points.

Again we have $x_u + x_{u'} = 0$ for each two mates u, u'. Similarly, we have inequalities as in (62), (66) and (68) if u and u' are outer end points.

Moreover, we have for each pair of inner end points u, u' belonging to one and the same L_i :

(127) $$x_u + x_{u'} \leq \beta_{u,u'}$$

where

(128) $$\beta_{u,u'} := min\ \{\varphi(D)\ |D \text{ is a curve in } \mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p) \text{ connecting } u \text{ and } u', \text{ homotopic to some curve in } L_i\}.$$

Again, this gives us a system $Ax \leq b$ of linear inequalities satisfying (69). Hence we can solve it in integers in polynomial time. The integer values are called the *shift numbers*. We shift each C_{ij} according to these shift numbers. After this shift we obtain curves $C'_{ij} (i = 1, \ldots, k; j = 1, \ldots, t_i)$ so that for each $i = 1, \ldots, k$ the closed curve

(129) $$K'_i := C'_1 \cdot C'_2 \cdot \ldots \cdot C'_{t_i}$$

does not enclose any $I_1, \ldots, I_p$, and so that no two different K'_i share the same disk D_v. Each K'_i gives in G a cycle K''_i so that two different K''_i are vertex-disjoint. Taking an arbitrary tree T'_i in K''_i spanning W_i gives a solution of problem (123).

Fixed Number of Holes

We can derive an extension of Theorem 34. Consider the problem:

(130) Given: – a graph $G = (V, E)$,
– sets $W_1, \ldots, W_k$ of vertices of G,
find: – pairwise vertex-disjoint trees $T_1, \ldots, T_k$ in G so that the vertex set of T_i contains W_i (for $i = 1, \ldots, k$).

This problem clearly is NP-complete, as the case $|W_1| = \ldots = |W_k| = 2$ is just the disjoint paths problem. Problem (130) is important to solve in VLSI-layout - it means that we must connect several sets of pins by pairwise disjoint interconnections.

Now the following can be derived from Theorem 36:

Theorem 37. *For each fixed p, there exists a polynomial-time algorithm for problem* (130) *if G is planar and* $W_1 \cup \ldots \cup W_k$ *can be covered by the boundaries of at most p faces of G.*

The idea is again to enumerate all "possibly feasible" choices of homotopies of trees $T_1, \ldots, T_k$ covering $W_1, \ldots, W_k$, respectively, similar to that used in deriving Theorem 34 from Theorem 31.

6. Edge-Disjoint Homotopic Paths

We finally consider the problem:

(131) Given: – a planar graph $G = (V, E)$ embedded in $\mathbb{R}^2$,
– faces $I_1, \ldots, I_p$ of G (including the unbounded face),
– curves $C_1, \ldots, C_k$ with end points on $bd\ (I_1 \cup \ldots \cup I_p)$,
find: – pairwise edge-disjoint paths $P_1, \ldots, P_k$ where P_i is
– homotopic to C_i in $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ $(i = 1, \ldots, k)$.

(Here "pairwise edge-disjoint" is assumed to include that no path uses the same edge twice.) This problem is NP-complete, as was shown by Kaufmann and Maley [17]. A main difference with the vertex-disjoint case is that for the edge-disjont case the given curves $C_1, \ldots, C_k$ might necessarily cross, so that the natural ordering of the curves in the vertex-disjoint case does not occur.

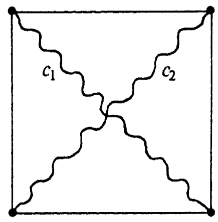
Clearly, a necessary condition for the solvability of (131) is:

(132) (*Cut condition*) for each curve D in $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ with end points on $bd\ (I_1 \cup \ldots \cup I_p)$ and not intersecting V one has:

$$cr\ (G, D) \geq \sum_{i=1}^{k} mincr\ (C_i, D).$$

This condition is not sufficient, as is shown by a very simple example:

(133)



So this gives no hope for obtaining interesting special cases where the cut condition is sufficient. However, under a parity condition, the problem turns out better to handle:

(134) (*Local parity condition*) for each $v \in V$:
$deg\ (v) + |\{i \in \{1, \ldots, k\} | C_i \text{ begins at } v\}| + |\{i \in \{1, \ldots, k\} \mid C_i \text{ ends at } v\}|$ is even.

More general is the following condition:

(135) *(Global parity condition)* for each curve D in $\mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$, with end points on $bd(I_1 \cup \ldots \cup I_p)$, not intersecting V and not touching edges, one has:

$$cr\ (G, D) \equiv \sum_{i=1}^{k} mincr\ (C_i, D) \quad (mod\ 2).$$

It is not difficult to derive (134) from (135). Kaufmann and Maley [17] showed that even under the local parity condition (134), problem (131) is NP-complete. It is not known whether this is also the case under the global parity condition (135). It turns out that the cut condition and one of the parity conditions are sufficient in some special cases.

Theorem 38. *If $p \leq 2$ and the local parity condition is satisfied, then problem* (131) *has a solution if and only if the cut condition* (132) *is satisfied.*

For $p = 1$ this is just the Okamura-Seymour theorem (Theorem 18). For $p = 2$, this is shown by Van Hoesel and Schrijver [10]. They also gave a polynomial-time method.

Theorem 39. *If*

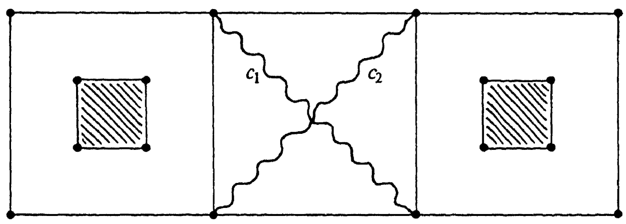
(136) *(i)* *G is part of the rectangular grid,*
(ii) *each face of G of area larger than 1 belongs to $I_1, \ldots, I_p$,*
(iii) *each vertex of degree 4 incident to exactly one face in $I_1, \ldots, I_p$ is not an end point of any of the curves $C_1, \ldots, C_k$,*
(iv) *the global parity condition holds,*

then: problem (131) *has a solution, if and only if the cut condition* (132) *holds.*

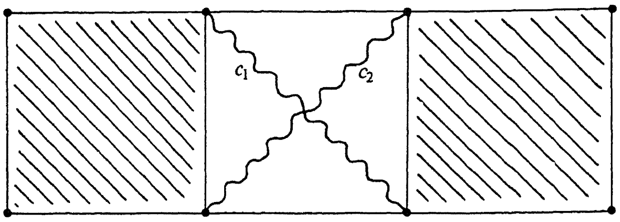
This was shown by Kaufmann and Melhorn [18], who also gave a polynomial-time algorithm for the corresponding problem. For an extension to "straight-line" planar graphs, see [41].

Example (136) shows that we cannot extend Theorem 38 to the case $p = 3$ (even if we assume the global parity condition):

(137)



Example (138) shows that in Theorem 39 it is not sufficient to assume just the local parity condition instead of the global parity condition:

(138) 

In fact, Kaufmann and Maley [17] showed that problem (131) is NP-complete if (136) holds with (iv) replaced by the local parity condition.

Although no solution exists in (137) and (138), there exists a "fractional" solution: we can find paths $P_1' \sim C_1, P_1'' \sim C_1, P_2' \sim C_2$, $P_2'' \sim C_2$ and scalars $\lambda_1' = \lambda_1'' = \lambda_2' = \lambda_2'' = \frac{1}{2}$ so that for each edge e:

(139) $$\lambda_1'\chi^{P_1'}(e) + \lambda_1''\chi^{P_1''}(e) + \lambda_2'\chi^{P_2'}(e) + \lambda_2''\chi^{P_2''}(e) \leq 1.$$

It turns out that, for any number of holes, the existence of such a fractional solution is equivalent to the cut condition, as was shown in [40]:

Theorem 40. *Let $G = (V, E)$ be a planar graph embedded in $\mathbb{R}^2$. Let $I_1, \ldots, I_p$ be some of the faces of G, including the unbounded face. Let $P_1, \ldots, P_k$ be paths in G with end points on the boundary of $I_1 \cup \ldots \cup I_p$. Then there exist paths $P_{11}, \ldots, P_{1t_1}, P_{21}, \ldots, P_{2t_2}, \ldots, P_{k1}, \ldots, P_{kt_k}$ in G and rationals $\lambda_{11}, \ldots, \lambda_{1t_1}, \lambda_{21}, \ldots, \lambda_{2t_2}, \ldots, \lambda_{k1}, \ldots, \lambda_{kt_k} \geq 0$ so that:*

(140) (i) $P_{ij} \sim P_i$ in $\mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$ $(i = 1, \ldots, k; j = 1, \ldots, t_i)$,

(ii) $\sum_{j=1}^{t_i} \lambda_{ij} = 1$ $(i = 1, \ldots, k)$,

(iii) $\sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_{ij}\chi^{P_{ij}}(e) \leq 1$ $(e \in E)$,

if and only if the cut condition (132) *is satisfied.*

Note that the λ_{ij} being integer would give a solution of (131).

Since the λ_{ij} can be found in polynomial time, with the help of the ellipsoid method (cf. [9]), we have as a consequence:

Theorem 41. *The cut condition* (132) *can be tested in polynomial time.*

We finally sketch the proof of Theorem 40. It is convenient to transform the space $\mathbb{R}^2 \backslash (I_1 \cup \ldots \cup I_p)$ into a compact orientable surface S : for each curve C_i, connecting I_j and $I_{j'}$ say, we add a "handle" between I_j and $I_{j'}$ and make C_i into a closed curve C_i' over this handle. Moreover, we extend the graph with an edge over the handle connecting the two end points of C_i. We do this for each C_i. In this way we obtain a compact orientable surface S. Then Theorem 40 follows from the following "homotopic circulation theorem":

Theorem 42. *Let $G = (V, E)$ be a graph embedded on a compact orientable surface S. Let $C_1, \ldots, C_k$ be closed curves on S. Then there exist cycles $B_{11}, \ldots, B_{1t_1}, \ldots, B_{k1}, \ldots, B_{kt_k}$ in G and rationals $\lambda_{11}, \ldots, \lambda_{1t_1}, \ldots, \lambda_{k1}, \ldots, \lambda_{kt_k} \geq 0$ so that:*

$$\text{(141)} \qquad \text{(i)} \quad \sum_{j=1}^{t_i} \lambda_{ij} = 1 \qquad (i = 1, \ldots, k),$$

$$\text{(ii)} \quad \sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_{ij} \chi^{B_{ij}}(e) \leq 1 \qquad (e \in E),$$

if and only if for each closed curve D on S not intersecting V we have:

$$\text{(142)} \qquad cr\ (G, D) \geq \sum_{i=1}^{k} mincr\ (C_i, D).$$

A *cycle* in G is a sequence

$$\text{(143)} \qquad (v_0, e_1, v_1, e_2, v_2, \ldots, e_\ell, v_\ell),$$

where $v_0, \ldots, v_\ell$ are vertices, with $v_0 = v_\ell$, and where e_i is an edge connecting v_{i-1} and $v_i (i = 1, \ldots, \ell)$. We identify in the obvious way such a cycle with a closed curve on S.

In fact, if S is the torus, we can take the λ_{ij} to be integers - see [8].

Basic in proving Theorem 42 is the following:

Theorem 43. *Let G be an eulerian graph embedded on a compact orientable surface S. Then the edges of G can be decomposed into cycles $C_1, \ldots, C_t$ in such a way that for each closed curve D on S:*

$$\text{(144)} \qquad mincr\ (G, D) = \sum_{i=1}^{t} mincr\ (C_i, D).$$

Decomposing the edges into cycles $C_1, \ldots, C_t$ means that each edge occurs in exactly one of the C_i, while in each C_i all edges are different. Moreover, *mincr* $(G, D) :=$ *min* $\{cr(G, \tilde{D}) \mid \tilde{D}, S_1 \longrightarrow S \backslash V(G); \tilde{D} \sim D\}$.

Our proof for this theorem is quite long, and uses some classical theorems in topology of Baer [2], Brouwer [3], von Kerékjártó [19] and Poincaré [32].

We do not know if Theorem 43 also holds for all compact nonorientable surfaces. In fact, it holds for the projective plane, in which case it is equivalent to Lins' theorem (Theorem 27 above).

In order to derive Theorem 42 from Theorem 43, we first derive the following from Theorem 43, using the duality of graphs on surface:

Theorem 44. *Let $G = (V, E)$ be a bipartite graph embedded on a compact orientable surface S, and let $C_1, \ldots, C_k$ be cycles in G. Then there exist closed curves $D_1, \ldots, D_t : S_1 \longrightarrow S$ so that (i) no D_j intersects V, (ii) each edge of G is intersected by exactly one D_j and by that D_j only once, (iii) for each $i = 1, \ldots, k$:*

(145) $$minlength_G\ (C_i) = \sum_{j=1}^{t} mincr\ (C_i, D_j).$$

Here we denote for any cycle C in G :

(146) $$length_G\ (C) := \ell, \text{ if } C = (v_0, e_1, v_1, \ldots, e_\ell, v_\ell),$$
$$minlength_G\ (C) := min\ \{length_G(\tilde{C}) \mid \tilde{C} \sim C, \tilde{C} \text{ cycle in } G\}.$$

(Cycles $\tilde{C}$ and C are allowed to pass one edge several times.)

Proof. We can extend (the embedded) G to a bipartite graph L embedded on S, containing G as a subgraph, so that each face of L (i.e., component of $S \backslash L$) is simply connected (i.e., homeomorphic to $\mathbb{R}^2$). Let $d := max\ \{minlength_G\ (C_i) \mid i = 1, \ldots, k\}$. By inserting d new vertices on each edge of L not occuring in G, we obtain a bipartite graph H satisfying

(147) $$minlength_G\ (C_i) = minlength_H\ (C_i)$$

for $i = 1, \ldots, k$.

Consider a dual graph H^* of H on S. Since H is bipartite, H^* is eulerian. Hence by Theorem 43 the edges of H^* can be decomposed into cycles $D_1, \ldots, D_t$ so that for any closed curve C on S :

(148) $$mincr\ (H^*, C) = \sum_{j=1}^{t} mincr\ (D_j, C).$$

Now for each $i = 1, \ldots, k, mincr(H^*, C_i) = minlength_H(C_i) = minlength_G\ (C_i)$, and (145) follows. □

Using the polarity relation of convex cones in eulerian space we derive finally Theorem 42 from Theorem 44. Necessity of (142) being trivial, we only show sufficiency.

Suppose (142) is satisfied for each closed curve D not intersecting V. Let K be the convex cone in $\mathbb{R}^k \times \mathbb{R}^E$ generated by the vectors:

(149) $$(\epsilon_i : \chi^\Gamma) \quad (i = 1, \ldots, k; \Gamma \text{ cycle in } G \text{ with } \Gamma \sim C_i);$$
$$(\mathbf{0}; \epsilon_e) \quad (e \in E).$$

Here ϵ_i denotes the i-th unit bases vector in $\mathbb{R}^k$. Similarly, ϵ_e denotes the e-th unit basis vector in $\mathbb{R}^E$. $\mathbf{0}$ denotes the origin in $\mathbb{R}^k$.

Although (149) gives infinitely many vectors, K is finitely generated. This can be seen as follows. For each fixed i, call a cycle $\Gamma \sim C_i$ *minimal* if there is no cycle $\Gamma' \sim C_i$ with $\chi^{\Gamma'}(e) \leq \chi^\Gamma(e)$ for each edge e, and with strict inequality for at least one edge e. So the set $\{\chi^\Gamma \mid \Gamma \text{ minimal cycle with } \Gamma \sim C_i\}$ forms an antichain in $\mathbb{Z}_+^E$ and is therefore finite. Since we can restrict, for each $i = 1, \ldots, k$, the χ^Γ in (149) to those with Γ minimal, K is finitely generated.

What we must show is that the vector $(\mathbf{1}; \mathbf{1}) = (1, \ldots, 1; 1, \ldots, 1)$ belongs to K.

By Farkas' lemma, it suffices to show that for each vector $(p,b) \in \mathbb{Q}^k \times \mathbb{Q}^E$ with nonnegative inner product with each of the vectors (149), also the inner product with $(\mathbf{1};\mathbf{1})$ is nonnegative. So let $(p;b)$ have nonnegative inner product with each of (149). This is equivalent to:

(150) (i) $p_i + \sum_{e\in E} b(e)\chi^{\Gamma}(e) \geq 0$ $\quad (i = 1,\ldots,k;\Gamma$ cycle in G with $\Gamma \sim C_i)$;
(ii) $b(e) \geq 0$ $\quad (e \in E)$.

Without loss of generality, each entry in $(p;b)$ is an even integer. Let G' be the graph arising from G by replacing each edge e by a path of length $b(e)$ (that is, $b(e)-1$ new vertices are inserted on e, if $b(e) \geq 1; e$ is contracted if $b(e) = 0$). Each cycle C_i in G directly gives a cycle C_i' in G'. Then by (150) (i):

(151) $$-p_i \leq minlength_{G'}(C_i') \quad \text{for } i = 1,\ldots,k.$$

Since G' is bipartite, by Theorem 44, there exist closed curves $D_1,\ldots,D_t$ on S so that (i) each D_j intersects G' only in edges of G', (ii) each edge of G' is intersected by exactly one D_j and only once by that $D_{j'}$ and (iii) for each $i = 1,\ldots,k$:

(152) $$minlength_{G'}\ (C_i') = \sum_{j=1}^{t} mincr\ (C_i', D_j).$$

Note that (ii) is equivalent to:

(153) $$b(e) = \sum_{j=1}^{t} \chi^{D_j}(e)$$

for each edge e of G. Therefore, using (142), (151), (152) and (153):

(154) $$\sum_{e\in E} b(e) = \sum_{j=1}^{t}\sum_{e\in E} \chi^{D_j}(e) = \sum_{j=1}^{t} cr\ (G, D_j) \geq$$
$$\sum_{j=1}^{t}\sum_{i=1}^{k} mincr\ (C_i, D_j) = \sum_{i=1}^{k}\sum_{j=1}^{t} mincr\ (C_i, D_j) =$$
$$\sum_{i=1}^{k} minlength_{G'}\ (C_i') \geq -\sum_{i=1}^{k} p_i.$$

So $(p;b)\cdot\ (\mathbf{1};\mathbf{1})\ ^T \geq 0$. □

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